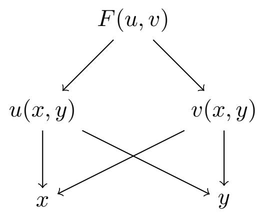
2. (25 Points) Consider a circle of radius R centered on the origin. You need to determine the coordinates of the points on the circle closest and farthest to a point outside the circle located at $P_0(\alpha, \beta)$. You may assume that P_0 is in the first quadrant. Clearly, one could construct a line from P_0 to the center, and then move a distance R along the line in either direction from the center. But, of course, this is *not* how we want you to solve the problem. As a Calculus III student, you need to impress your graders by doing this calculation using Calculus III concepts. The higher the concept level, the higher your possible grade. Of course you will also justify and explain your reasoning as you progress through this problem. Right?

   Note: you were in diapers when this problem was used. Guaranteed.

3. (25 Points) Consider the function $F(u, v)$, where u and v are functions of x and y specifically, $u(x, y)$ and $v(x, y)$ respectively. For a particular set of F, x, y, u, and v values, $F_u = 1$, $F_v = \beta$, $u_x = \alpha$, $u_y = 2$, $v_x = 2$, $v_y = 3$, where α and β are real constants. Reread all of this to make sure you've got it all straight. Maybe one or two more times, just to make sure.



(a) Suppose you are now told that for the above conditions, $dF = 7\,dx + 8\,dy$. If x changes by the small amount 0.01 and y changes by the small amount -0.02, estimate the change in the value of F.

(b) If $dF = 7\,dx + 8\,dy$ still holds, then determine the values of α and β.

(c) When x changes by the small amount 0.01, and y changes by the small amount -0.02, estimate the change in the value of u.

(d) When x changes by the small amount 0.01, and y changes by the small amount -0.02, estimate the change in the value of v.

(e) Ultimately thinking of F as a function of x and y, and if possible, determine $\nabla F = F_x\,\mathbf{i} + F_y\,\mathbf{j}$ for the conditions described in the previous parts of the problem. Otherwise clearly state "Cannot be determined."

4. (25 Points) Consider the function $f(x, y) = \exp(x + y)$. Warning: carefully read all the numeric values in this question, then read them again.

   (a) Calculate the *second order* Taylor approximation to $f(x, y)$ near the point $(2, 1)$.

   (b) Use your result from part (a) to estimate the value of $f(2.2, 1.1)$. Do not simplify your answer here. For example, you can leave your answer in the form $8 + 4(3.1 - 3) + 3(4.01 - 4)$, although we really do not recommend using these numbers.

   (c) Calculate an "upper bound on the error" associated with your *second order* approximation assuming that you only use values of x and y such that $|x - 2| \leq 0.2$ and $|y - 1| \leq 0.1$. Please simplify your answer, but don't try to convert to decimal form.

   (d) Now suppose you actually worked out the *fifth order* Taylor approximation to $f(x, y)$ near the point $(3, 2)$. (You don't actually need to work out this approximation! Also note the change in the center location from $(2, 1)$ to $(3, 2)$.) Calculate an "upper bound on the error" associated with this *fifth order* approximation assuming that you only use values of x and y such that $|x - 3| \leq 0.1$ and $|y - 2| \leq 0.1$. Please simplify your answer, but don't try to convert to decimal form.

Projections and distances

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + \Big[\,(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\ &\quad + \frac{1}{2!}\Big[\,(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\ &\quad + \frac{1}{3!}\Big[\,(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ &\qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{M}{2}(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{\,|f_{xx}|,\,|f_{xy}|,\,|f_{yy}|\,\} \le M$$